

SEMESTER - I
INTRODUCTION TO MATHEMATICAL PHYSICS
UNIT - I : VECTOR ANALYSIS



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SEMESTER-I
COURSE 1: INTRODUCTION TO MATHEMATICAL PHYSICS

Theory Credits: 3 3 hrs/week

COURSE OBJECTIVE:

To equip students with foundational mathematical techniques—such as vector calculus, linear algebra, complex numbers, probability, and Fourier analysis—essential for understanding and solving problems in physics.

LEARNING OUTCOMES:

After successful completion of the course, students will be able to:

1. Apply concepts of vector differentiation and integration to analyze physical fields and prove integral theorems.
2. Use matrix operations and eigenvalue techniques to solve linear systems in physics.
3. Represent and manipulate complex numbers in various forms for solving AC circuit problems.
4. Interpret and apply basic probability concepts and distributions to model physical phenomena.
5. Analyze periodic signals using Fourier series and evaluate Fourier coefficients for common waveforms.

UNIT-I - VECTOR ANALYSIS (9. Hrs.)

Distinction between Ordinary and partial derivatives, Scalar and vector fields, gradient of a scalar field and its physical significance. Divergence and curl of a vector field with derivations and physical interpretation. Vector integration (line, surface and volume), Statement and proof of Gauss and Stokes theorems.

UNIT-II – LINEAR ALGEBRA (9. Hrs.)

Vector and Scalar quantities in Physics, Matrices and Operations: Types, Addition and Multiplication, Identity and Inverse, Determinant (2x2 and 3x3), Trace, Transpose, Eigenvalues and Eigen Vectors, Calculation of Eigen values using characteristic equations. System of Linear Equations: Solving 2-variable system using matrices, Simple examples from physics (Current, forces)

UNIT – III COMPLEX NUMBERS (9. Hrs.)

Basic Complex numbers: Real and imaginary parts, Conjugate of complex numbers, Modulus and argument (magnitude and phase), Polar and Exponential (Euler) form of complex numbers. Addition and subtraction of complex numbers, Multiplication and division of complex numbers. Phasor representation: representation of voltage and current as phasors, Derivation of Impedance of a series LCR circuit.

UNIT – IV PROBABILITY (9. Hrs.)

Probability Theory Basics, Sample space, events, conditional probability, and Bayes' theorem. Independence and mutual exclusivity. Random Variables and Probability Distributions, Concept of random variables (discrete and continuous). Common distributions and their applications: Binomial, Poisson, and Gaussian.

UNIT V FOURIER ANALYSIS (9. Hrs.)

Introduction to periodic functions: Concept of periodicity (waves, oscillations, AC current), Graphical understanding of Sine and Cosine functions, Need for Fourier analysis, Real world signals (heartbeat, electrical signal, musical tones), Fourier theorem and evaluation of Fourier coefficients, Analysis of periodic wave functions – Square wave, saw tooth wave and triangular wave.

Reference books

1. Mathematical methods for physics sciences (3rd edition) - Mary. L. Boas
2. First Chapter (Vector analysis) in Introduction to Electrodynamics (3rd edition) – David. J. Griffiths
3. Mathematical Methods for Physicists: Arfken, Weber, 2005, Harris, Elsevier

Student Activities:

- Problem-solving sessions using real-life physics applications (e.g., using vector calculus in electromagnetism).
- Group projects on solving physical systems using matrix methods and linear algebra tools.
- Mini-lab activity on phasor diagrams and impedance using circuit simulation software (like LTspice or Tinkercad Circuits).
- Data collection and analysis task: Record physical measurements (e.g., decay times, counts) and apply statistical models (Poisson/Gaussian)

SEMESTER-I
COURSE 1: INTRODUCTION TO MATHEMATICAL PHYSICS

Practical Credits: 1 2 hrs/week

COURSE OBJECTIVE:

To develop foundational computational and analytical skills through hands-on exercises that prepare students for understanding and solving problems in various realms of physics.

LEARNING OUTCOMES:

1. Graphing and Visualization:

Students will be able to plot mathematical functions and visualize physical phenomena using Excel, Python, or MATLAB.

2. Vector and Matrix Computations:

Students will perform operations on vectors and matrices and represent them both analytically and graphically.

3. Numerical Methods:

Students will apply numerical techniques like Newton-Raphson, Bisection, and Euler's method to solve equations and differential equations.

4. Data Analysis and Fitting:

Students will analyze experimental data using tools like least squares fitting and compute statistical quantities such as mean, standard deviation, and error.

5. Fourier and Complex Number Representation:

Students will approximate functions using Fourier series and graphically represent complex numbers.

List of Practicals

Minimum of 6 experiments to be conducted and recorded

1. Graphing standard functions: $\sin(x)$, $\cos(x)$, e^x , $\ln(x)$, x^2 , \sqrt{x} etc. using Excel/Python/Graph paper
2. Experimental determination and vector diagram verification of vector addition and scalar product using graphical methods.
3. Using MATLAB/Python to visualize vector fields and compute gradient, divergence, and curl.
4. Solve simple non-linear equations (e.g., $x^3 - x - 1 = 0$) using graphical methods and bisection/newton-raphson method (Python or Excel).
5. Fit experimental data (e.g., Hooke's law) to a straight line using least squares method in Excel or Python.
6. Linear equation Solution and System of linear equation solution using MATLAB/OCTAVE
7. Fourier approximation of a square wave up to 5 terms using Python/MATLAB and plotting the result.
8. Numerical solution of $dy/dx = x + y$, given initial condition using Euler's method.
9. Single coin tossing and four coin tossing using MATLAB/OCTAVE and verification of statistical laws
10. Use Python/Excel to perform addition, multiplication, and finding inverse of 2×2 and 3×3 matrices.
11. Simulate and plot s-t, v-t graphs using $s = ut + 0.5gt^2$ using Excel or Python.
12. Calculate mean, standard deviation, and percentage error for a given data set using Excel/Python/Manual calculations
13. Represent the given complex numbers on graph paper
14. Determine the Eigen Values of the given matrix using characteristic equation

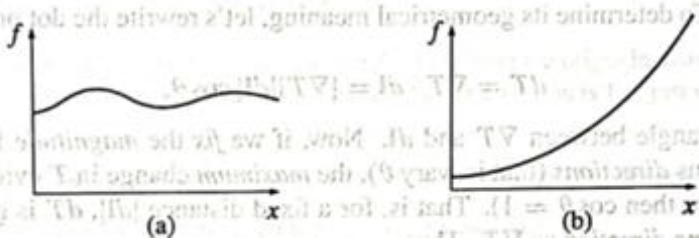
1.2.1 “Ordinary” Derivatives

Question: Suppose we have a function of one variable: $f(x)$. What does the derivative, df/dx , do for us? **Answer:** It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx :

$$df = \left(\frac{df}{dx} \right) dx. \quad (1.33)$$

In words: If we change x by an amount dx , then f changes by an amount df ; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x , and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x , and the derivative is large, as you move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the slope of the graph of f versus x .



Partial Differentiation

face (see Figure 1.1). The points satisfying $z = f(x, y)$ and $x = \text{const.}$ then lie on a curve (the curve of intersection of the surface and the $x = \text{const.}$ plane; this is AB in Figure 1.1). We might want the slope, maximum and minimum points, etc., of this curve. Since z is a function of y (on this curve), we might write dz/dy for the slope. However, to show that z is actually a function of two variables x and y with one of them (x) temporarily a constant, we write $\partial z/\partial y$; we call $\partial z/\partial y$ the partial derivative of z with respect to y . Similarly, we can

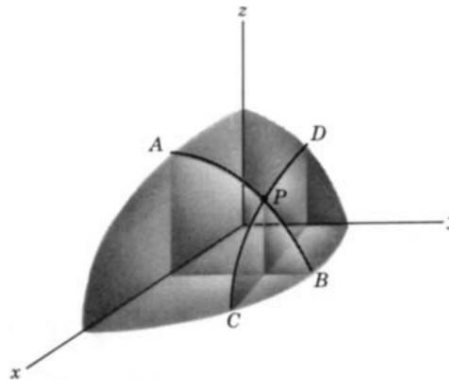


Figure 1.1

hold y constant and find $\partial z/\partial x$, the partial derivative of z with respect to x . If these partial derivatives are differentiated further, we write

$$\frac{\partial}{\partial x} \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial}{\partial x} \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial}{\partial x} \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^3 z}{\partial x^2 \partial y}, \quad \text{etc.}$$

Other notations are often useful. If $z = f(x, y)$, we may use z_x or f_x or f_1 for $\partial f/\partial x$, and corresponding notations for the higher derivatives.

► **Example.** Given $z = f(x, y) = x^3 y - e^{xy}$, then

$$\begin{aligned} \frac{\partial f}{\partial x} &\equiv \frac{\partial z}{\partial x} \equiv f_x \equiv z_x \equiv f_1 = 3x^2 y - y e^{xy}, \\ \frac{\partial f}{\partial y} &\equiv \frac{\partial z}{\partial y} \equiv f_y \equiv z_y \equiv f_2 = x^3 - x e^{xy}, \\ \frac{\partial^2 f}{\partial x \partial y} &\equiv \frac{\partial^2 z}{\partial x \partial y} \equiv f_{yx} \equiv z_{yx} \equiv f_{21} = 3x^2 - e^{xy} - xy e^{xy}, \\ \frac{\partial^2 f}{\partial x^2} &\equiv \frac{\partial^2 z}{\partial x^2} \equiv f_{xx} \equiv z_{xx} \equiv f_{11} = 6xy - y^2 e^{xy}, \\ \frac{\partial^3 f}{\partial y^3} &\equiv \frac{\partial^3 z}{\partial y^3} \equiv f_{yyy} \equiv z_{yyy} \equiv f_{222} = -x^3 e^{xy}, \\ \frac{\partial^3 f}{\partial x^2 \partial y} &\equiv \frac{\partial^3 z}{\partial x^2 \partial y} \equiv f_{yxx} \equiv z_{yxx} \equiv f_{211} = 6x - 2y e^{xy} - xy^2 e^{xy}. \end{aligned}$$

Scalar and Vector fields :

Scalar field:

A scalar field assigns a single number (magnitude) to every point in space. A point function $\Phi(x,y,z)$ gives different values depending on x,y and z coordinates produces scalar field in the range of variables. Examples: Temperature or Electric potential.

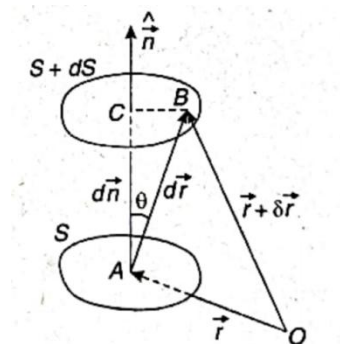
Vector field:

A vector field assigns a vector (magnitude and direction) to every point in space. Here the point function $\Phi(x,y,z)$ represents a vector quantity in a range of variables x, y and z. The vector lines of a vector field are called as stream lines or flux lines. No. of flux lines crossing unit area in a vector field normal to the direction of a vector is called as vector flux. Example : Fluid velocity field or Electric field

Gradient of a Scalar field :

The gradient of a scalar field is a vector field whose magnitude is equal to the maximum rate of change and direction is along the normal drawn to the surface

Let us consider two circles S and S+dS which are very close to each other as shown in the diagram



Here A, B are two parts on the above surfaces. Let us consider a point 'O' outside the surfaces as shown in the diagram.

A radius vector r is drawn from O to A. Similarly another radius vector r+dr is drawn from O to B. The normal AC is drawn as shown in the diagram. \hat{n} is the unit vector perpendicular to the surface S.

The rate of change of S at point A = $\frac{\partial S}{\partial n}$

$$ds = \frac{\partial S}{\partial n} \cdot dn$$

$$ds = \frac{\partial S}{\partial n} dr \cos \theta$$

From figure $ds = \left(\frac{\partial S}{\partial n}\right) \hat{n} \cdot dr$

Here the quantity $\left(\frac{\partial S}{\partial n}\right) \hat{n}$ is called the gradient of scalar quantity. Which is the gradient of S

Hence the gradient of scalar field is a vector quantity.

Physical Significance : We can convert the scalar physical quantities like gravitational and electric potential into vectors.

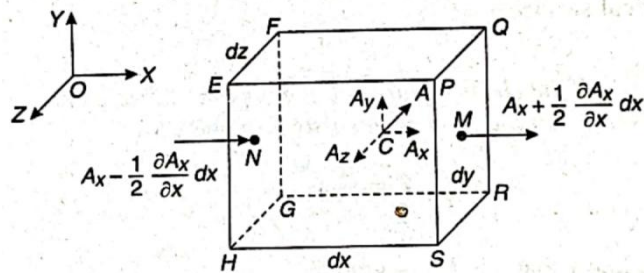
The gradient of the scalar field can be defined as a vector. Using gradient we can associate a scalar with a vector.

Divergence of a Vector field :

The divergence of a vector field at any point is defined as the amount of flux per unit volume diverging from that point

It can be expressed as $\nabla \cdot \mathbf{A} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot (iA_x + jA_y + kA_z)$
 $\nabla \cdot \mathbf{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$

Proof : Let us consider a small volume element of volume dv as shown in the diagram



Flux entering into the volume along X-direction is $\left(A_x - \frac{1}{2} \frac{\partial A_x}{\partial x} dx \right) dydz$

Similarly flux leaving the volume element along X-direction is $\left(A_x + \frac{1}{2} \frac{\partial A_x}{\partial x} dx \right) dydz$

Excess of flux leaving the volume element along X-direction $\frac{\partial A_x}{\partial x} dx dy dz$

Excess of flux leaving the volume element along Y-direction $\frac{\partial A_y}{\partial y} dx dy dz$

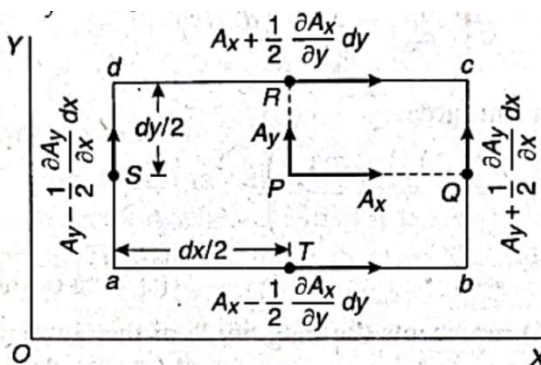
Similarly excess of flux along Z-direction $\frac{\partial A_z}{\partial z} dx dy dz$

Total flux diverging from the volume $= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dx dy dz$

Amount of flux diverging from unit volume $\text{div } \mathbf{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$

Curl of a vector field : This is also known as rotation of a vector

Definition : The curl of any vector \mathbf{A} is the maximum line integral per unit surface area



$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Proof : Let us consider surface area abcd as shown in the diagram

The rate of increase of vector component \mathbf{A}_y along X-direction is $\frac{\partial A_y}{\partial x}$

Similarly the rate of increase of vector component \mathbf{A}_x along Y-direction is $\frac{\partial A_x}{\partial y}$

Value of \mathbf{A}_x at R as we move from P to R $\left(A_x + \frac{dy}{2} \frac{\partial A_x}{\partial y} \right)$

Similarly value of \mathbf{A}_y at point Q $= \left(A_y + \frac{dx}{2} \frac{\partial A_y}{\partial x} \right)$

At point S value of \mathbf{A}_y (component) $= \left(A_y - \frac{dx}{2} \frac{\partial A_y}{\partial x} \right)$

At point T value of $\mathbf{A}_x = \left(A_x - \frac{dy}{2} \frac{\partial A_x}{\partial y} \right)$

Maximum line integral along the boundary abcd is given by

$$= (\text{ab}) \times (\text{component along ab}) + (\text{bc}) \times (\text{component along bc}) + (\text{cd}) \times (\text{component along cd}) + (\text{da}) \times (\text{component along da})$$

$$= \left(A_x - \frac{dy}{2} \frac{\partial A_x}{\partial y} \right) dx + \left(A_y + \frac{dx}{2} \frac{\partial A_y}{\partial x} \right) dy - \left(A_x + \frac{dy}{2} \frac{\partial A_x}{\partial y} \right) dx - \left(A_y - \frac{dx}{2} \frac{\partial A_y}{\partial x} \right) dy$$

$$= \frac{\partial A_y}{\partial x} dx dy - \frac{\partial A_x}{\partial y} dx dy$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy$$

The maximum line integral per unit area $= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$

The above equation is the magnitude of Curl A along Z - direction

$$\text{Curl}_z A = k \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\text{Similarly } \text{Curl}_y A = j \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)$$

$$\text{Curl}_x A = i \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)$$

$$\text{Curl } \mathbf{A} = \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

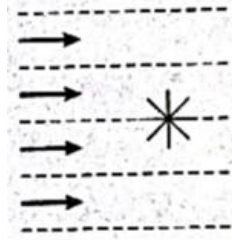
In terms of ∇ and \mathbf{A}

$$\nabla = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \quad \text{and} \quad \mathbf{A} = (iA_x + jA_y + kA_z) \quad \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

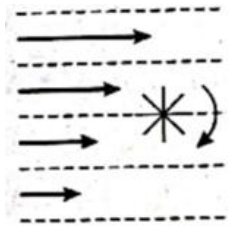
Physical significance of curl :

Curl means rotation. Let us consider different layers of the liquid moving at constant speed

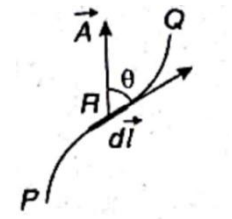


Here the paddle remains stationary or do not move

Let us consider a liquid flow here different layers of the liquid will be moving with different velocities as shown in the diagram



Line integral :



Let us consider a curved line (PQ) as shown in the diagram.

We are moving from P to Q

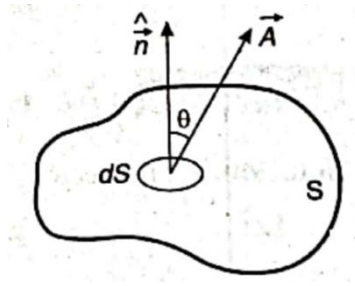
Consider a small line segment $d\mathbf{l}$ as shown in the diagram

The vector \mathbf{A} makes an angle θ with $d\mathbf{l}$. Component of \mathbf{A} along $d\mathbf{l}$ is given by the dot product $\mathbf{A} \cdot d\mathbf{l}$. The line integral is given by

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l}$$

Surface integral :

Let us consider a surface S as shown in the figure



Let us consider a small surface area $d\mathbf{s}$ as shown in the diagram. Here $\hat{\mathbf{n}}$ is a unit vector normal to the surface area $d\mathbf{s}$. \mathbf{A} is the vector field which is at an angle θ with the $\hat{\mathbf{n}}$

Component of \mathbf{A} along $\hat{\mathbf{n}}$ is $\mathbf{A} \cdot \hat{\mathbf{n}}$

The area vector $d\mathbf{s}$ is given by $d\mathbf{s} = (\mathbf{A} \cdot \hat{\mathbf{n}})d\mathbf{s}$

The total surface area vector \mathbf{S} is given by integrating the above expression

$$\mathbf{S} = \iint (\mathbf{A} \cdot \hat{\mathbf{n}})d\mathbf{s} = \iint A \cos \theta d\mathbf{s} = \iint \mathbf{A} \cdot d\mathbf{s}$$

Volume integral :

Let us consider a big volume \mathbf{V} in which a small volume element $d\mathbf{v}$ is considered. Let \mathbf{A} be a vector on any point of the volume element's surface. Then the volume vector is $\mathbf{A} \cdot d\mathbf{V}$. The total volume is given by integrating the above expression three times

$$V = \iiint \mathbf{A} \cdot d\mathbf{v}$$

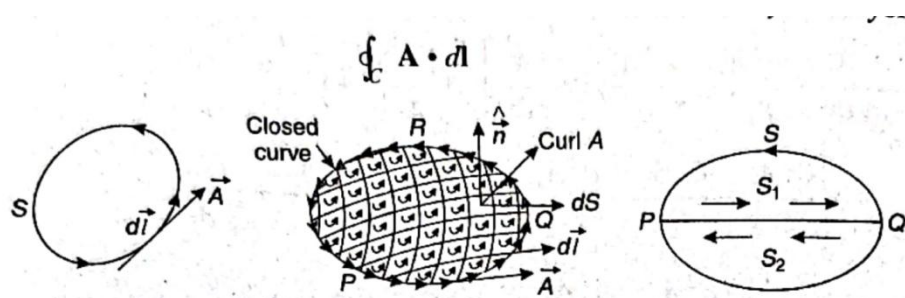
Stoke's theorem :

Definition : The line integral of the vector field \mathbf{A} around the closed curve S is equal to the surface integral of curl of \mathbf{A} taken over the surface S surrounded by the closed curve

Proof : Let us consider a closed curve S as shown in the diagram. Let \mathbf{A} be the vector field around the closed curve. Let $d\mathbf{l}$ be the small segment around this curve

Component of \mathbf{A} along $d\mathbf{l}$ is $(\mathbf{A} \cdot d\mathbf{l})$

The line integral around the curve = = $\oint \mathbf{A} \cdot d\mathbf{l}$ -----(1)



Let the entire area is divided into small square loops having area $d\mathbf{s}$. Let \hat{n} be the unit normal vector drawn in the middle of $d\mathbf{s}$. The angle between \mathbf{A} and \hat{n} is θ . The normal component of vector \mathbf{A} along \hat{n} is $A \cos \theta$ or $\mathbf{A} \cdot \hat{n}$. The normal component for the surface area $d\mathbf{s}$ is $(\mathbf{A} \cdot \hat{n})d\mathbf{s}$ or $(\mathbf{A} \cdot d\mathbf{s})$ from the definition of curl, curl \mathbf{A} is the maximum line integral around the boundary line of the closed surface $d\mathbf{s}$. The vector area on the surface $d\mathbf{s}$ is given by $\text{curl } \mathbf{A} \cdot d\mathbf{s}$. Hence the surface area over the entire surface S is given by $\iint \text{curl } \mathbf{A} \cdot d\mathbf{s}$ -----(2).

From figure the line integrals along the common sides of the continuous elements mutually cancel because they traverse the opposite directions. Now the sides of the element which lie in the periphery of the surface contribute to the line integral. The sum of line integrals on the boundary line of the curve is given by equation (2) which is also given by equation (1) hence (1) = (2)

$$\oint \mathbf{A} \cdot d\mathbf{l} = \iint \text{curl } \mathbf{A} \cdot d\mathbf{s}$$

This is called Stoke's theorem

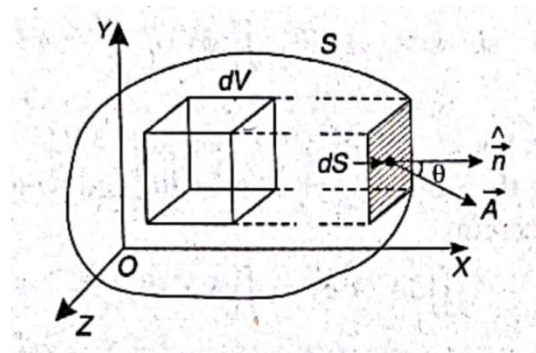
This theorem is useful in converting the surface integrals into line integrals vice-versa

Gauss Divergence theorem :

Definition : The surface integral of the normal component of vector \mathbf{A} taken over a closed surface \mathbf{S} is equal to the volume integral of the divergence of vector \mathbf{A} over the volume \mathbf{V} enclosed by the surface \mathbf{S}

$$\iint \mathbf{A} \cdot d\mathbf{s} = \iiint \text{div } \mathbf{A} \, dV = \iiint (\nabla \cdot \mathbf{A}) \, dV$$

Let us consider a closed surface \mathbf{S} as shown in the diagram below



Proof : Let the entire volume \mathbf{V} is divided into small volume elements each having a volume of dV . We know that $\text{div } \mathbf{A}$ represents amount of flux per unit volume. Hence the flux through the volume element dV is given by $\text{div } \mathbf{A} \cdot dV$

Flux through the entire volume is given by

$$\iiint \text{div } \mathbf{A} \, dV \text{ -----(1)}$$

The normal components of vector \mathbf{A} along \hat{n} is given by $(\mathbf{A} \cdot \hat{n})$

The normal components of vector field \mathbf{A} on the surface $d\mathbf{s}$ is given by

$$(\mathbf{A} \cdot \hat{n})ds = \mathbf{A} \cdot d\mathbf{s}$$

The normal components through the entire surface \mathbf{S} is given by

$$\iint \mathbf{A} \cdot d\mathbf{s} \text{ -----(2)}$$

This is nothing but flux through the entire surface

Since (1) and (2) represent same physical quantity (1) = (2)

$$\iint \mathbf{A} \cdot d\mathbf{s} = \iiint \text{div } \mathbf{A} \, dV$$

This is known as Gauss Divergence theorem. It is a kind of method using which we connect surface integrals into volume integrals.